BATES, ROBERT FRANK. Separation Axioms Weaker Than T_2. (1972)
Directed by: Dr. Hughes B. Hoyle, III pp. 21

A KC space is a space in which every compact subset is closed. A US space is a space in which every convergent sequence has a unique limit. In this paper, it is shown that every T_2-space is a KC space is a US space is a T_1-space and that no implication is reversible.

SEPARATION AXIOMS WEAKER THAN T_2

by

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A Thesis Submitted to
the Faculty of the Graduate School at
The University of North Carolina at Greensboro
in Partial Fulfillment
of the Requirements for the Degree
Master of Arts

Greensboro
February, 1972

Approved by

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ACKNOWLEDGMENT

The author wishes to express his appreciation to Dr. Hughes B. Hoyle for his patience and guidance during the preparation of this thesis.

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INTRODUCTION

The purpose of this thesis is to briefly investigate ideas of Wilansky [2] on separation axioms between T_1 and T_2.

In Chapter 1, preliminary definitions and theorems are stated and proved.

In Chapter 2, a KC space, a US space, and a K space are defined and the properties of KC and US are put in order between T_1 and T_2. Examples are given to show that no two ideas are in general equivalent.

The reader is expected to have a working knowledge of point set topology and is referred to [1], [2], and [3] for definitions and theorems not covered in this paper.

CHAPTER I

PRELIMINARY RESULTS

Definition 1: If (X, T) is a topological space and $x \in X$, then $V \subset X$ is a neighborhood of x if there exists an open set U with $x \in U \subset V$.

Definition 2: If (X, T) is a topological space and $x \in X$, then the collection Ψ_x of all neighborhoods of x is the neighborhood system at x.

Definition 3: If (X, T) is a topological space and $x \in X$, then a neighborhood base at x is a subcollection $\S_x$, taken from the neighborhood system Ψ_x, having the property that for each $U \in T$ with $x \in U$, then there is a $V \in \S_x$ such that $x \in V \subset U$.

Definition 4: If (X, T) is a topological space and Σ a collection of subsets of X whose union is X, then Σ is a cover of X. A subcover of a cover Σ is a subcollection Θ of Σ which is a cover of X. An open cover of X is a cover consisting of sets from T.

Definition 5: Given two sets X and Y, the cross product of X and Y, denoted by $X \times Y$, is the set of all ordered pairs (x, y) such that $x \in X$ and $y \in Y$.

Definition 6: A function f from a set X into a set Y, denoted by $f: X \to Y$, is a subset of $X \times Y$ such that if $x \in X$, then there exists some $y \in Y$ such that $(x, y) \in f$, and if

$(x, y) \in f$ and $(x, z) \in f$, then $y = z$. The set X is called the domain of f, and $\{y \mid (x, y) \in f\}$ is called the range of f.

Definition 7: A sequence S is a function whose domain is a subset of the set of positive integers such that 1 is an element of the domain of S, and if U is in the domain of S and V is a positive integer less than U, then V is in the domain of S.

Definition 8: A set is said to be countable provided it is the range of some sequence, and a set is said to be finite provided it is the range of a sequence whose domain does not contain all the positive integers. Also, a set is said to be uncountable provided it is not countable.

Definition 9: A topological space (X, T) is said to be compact provided each open cover of X has a finite subcover.

Definition 10: If (X, T) is a topological space, and A is a nonempty subset of X, then $T_A = \{A \cap U \mid U \in T\}$. A subset V of X is said to be T_A-open provided $V \in T_A$.

Theorem 1: If (X, T) is a topological space and A is a nonempty subset of X, then the ordered pair (A, T_A) is a topological space.

Proof: Since $\phi = \phi \cap A$ and $A = X \cap A$, then $\phi, A \in T_A$. Suppose $U, V \in T_A$. Then there are elements O_1 and O_2 in T such that $U = A \cap O_1$ and $V = A \cap O_2$.
Then $U \cap V = (A \cap O_1) \cap (A \cap O_2) = (O_1 \cap O_2) \cap A$. Thus $U \cap V \in T_A$. Let $\Psi = \{U_\alpha \mid \alpha \in \Gamma\}$ such that for each $\alpha \in \Gamma$, $U_\alpha \in T_A$. Now, if $\alpha \in \Gamma$, then there is an element $O_\alpha \in T$ such that

$U_\alpha = O_\alpha \cap A$. Then

$\cup\{U_\alpha \mid \alpha \in \Gamma\} = \cup\{O_\alpha \cap A \mid \alpha \in \Gamma\} = A \cap (\{\cup O_\alpha \mid \alpha \in \Gamma\})$. But $\cup\{O_\alpha \mid \alpha \in \Gamma\} \in T$. Thus $A \cap (\cup\{O_\alpha \mid \alpha \in \Gamma\}) \in T_A$, and hence $\cup\{U_\alpha \mid \alpha \in \Gamma\} \in T_A$. Hence T_A is a topology for A.

Definition 11: If (X, T) is a topological space and A is a nonempty subset of X, then T_A is said to be the relative topology on A. Also (A, T_A) is said to be a subspace of (X, T).

Theorem 2: Let (X, T) be a compact topological space and let A be a closed subset of X. Then (A, T_A) is compact.

Proof: Let Ψ be a T_A-open cover of A. For each $U \in \Psi$, there is a $V_U \in T$ such that $U = V_U \cap A$. Let $\Sigma = \{V_U \mid U \in \Psi\} \cup \{X - A\}$. Then Σ is a T-open cover of X, and hence Σ has a finite subcover $V_{U_1}, V_{U_2}, \ldots, V_{U_N}$, and possibly $X - A$. Then $U_1, U_2, \ldots, U_N$ is a finite subcover of Ψ and hence (A, T_A) is compact.

Definition 12: If (X, T) is a topological space, then (X, T) is said to be a T_2-space provided if $a, b \in X$ with $a \neq b$, then there exist $U, V \in T$ such that $a \in U$, $b \in V$, and $U \cap V = \phi$.

Theorem 3: If (X, T) is a T_2 topological space, and if (A, T_A) is a subspace of (X, T), then (A, T_A) is a T_2-space.

Proof: Let $a, b \in A$ with $a \neq b$. Then there exist disjoint open sets U and V in T containing a and b respectively. Now $U \cap A \in T_A$, $V \cap A \in T_A$, $a \in U \cap A$, and $b \in V \cap A$. Also $(U \cap A) \cap (V \cap B) = \phi$. Thus (A, T_A) is a T_2-space.

Definition 13: If (X, T) is a topological space, then (X, T) is said to be regular provided if A is a closed set in X and $x \in X - A$, then there exist disjoing sets U and V in T such that $A \subset U$ and $x \in V$.

Theorem 4: Let (X, T) be a T_2 topological space. Let A be a subset of X such that (A, T_A) is compact, and let $x \in X - A$. Then there exist sets $U, V \in T$ such that $x \in V$ and $A \subset U$ and $U \cap V = \phi$.

Proof: Let $a \in A$. Since (X, T) is a T_2-space, there exist disjoint sets U_a, $V_a \in T$ such that $x \in V_a$ and $a \in U_a$. Let $W_a = U_a \cap A$. Then $\{W_a \mid a \in A\}$ is a T_A-open cover of A. Since (A, T_A) is compact, there is a finite subcover $W_{a_1}, W_{a_2}, \ldots, W_{a_n}$ of $\{W_a \mid a \in A\}$. Let $U = \bigcup_{i=1}^{n} U_{a_i}$ and $V = \bigcap_{i=1}^{n} V_{a_i}$. Then $A \subset U$, $x \in V$ and $U \cap V = \phi$.

Theorem 5: Let (X, T) be a T_2 topological space, and let A be a subset of X such that (A, T_A) is compact. Then A is a closed subset of X.

Proof: Suppose A is not closed. Then there is a limit point p of A which is not in A. By Theorem 4, there exist disjoint sets $U, V \in T$ such that $p \in V$ and $A \subset U$. But this is impossible since p is a limit point of A. Hence A is closed in X.

Definition 14: Let (X, T) be a topological space. Then (X, T) is a T_1-space provided if $a, b \in X$ with $a \neq b$, then there exists a set $U \in T$ containing a but not b, and there exists a set $V \in T$ containing b but not a.

Theorem 6: Let X be an uncountable set. Let T be the collection of subsets of X to which U belongs provided either $U = \phi$ or $X - U$ is finite. Then T is a topology for X.

Proof: Clearly ϕ and X are elements of T. Suppose $U, V \in T$. Then $X - U$ and $X - V$ are finite. So $(X - U) \cup (X - V)$ is finite, and $(X - U) \cup (X - V) = X - (U \cap V)$ by DeMorgan's Laws. Thus $X - (U \cap V)$ is finite and $U \cap V \in T$. Let $\S = \{U_\alpha \mid \alpha \in \Gamma\}$ such that if $\alpha \in \Gamma$, then $U_\alpha \in T$. Now, let $\beta \in \Gamma$. Then $X - U_\beta$ is finite. Since $U_\beta \subset \cup\{U_\alpha \mid \alpha \in \Gamma\}$, it follows that $X - \cup\{U_\alpha \mid \alpha \in \Gamma\} \subset X - U_\beta$. Then $X - \cup\{U_\alpha \mid \alpha \in \Gamma\}$ must be finite.

Definition 15: The topology T in Theorem 6 is called the cofinite topology for X.

Example 1: Let X be an uncountable set. Let T be the cofinite topology for X, and let $a \in X$. Then (X, T) is a T_1-space which is not a T_2-space, and $X - \{a\}$ is a compact subset of (X, T) which is not closed.

Proof: Let $b \in X$ with $a \neq b$. Then $U_b = X - \{b\}$ is an open set containing a but not b. Also, $U_a = X - \{a\}$ is an open set containing b but not a. Therefore (X, T) is a T_1-space. Since any two open sets in X intersect, (X, T) is not a T_2-space. Finally, let $\S$ be an open cover of $X - \{a\}$ by $T_{X - \{a\}}$ open sets. Let $V \in \S$. Then V contains all but a finite number of points of $X - \{a\}$. If $x_i \in X$ such that $x_i \notin V$, then let U_{x_i} be an element of $\S$ such that $x_i \in U_{x_i}$. Then

$\Sigma = V \cup \{U_{x_1} \cup U_{x_2} \cup \ldots \cup U_{x_n}\}$ is a finite subcover of $\S$

by $T_{X - \{a\}}$ open sets. Thus $X - \{a\}$ is a compact subset of (X, T). Since $\{a\} \notin T$, $X - \{a\}$ is not closed.

<u>Theorem 7</u>: Let (X, T) be a regular topological space, let $x \in X$, and let $U \in T$ such that $x \notin U$. Then there exists a $V \in T$ such that $x \in V \subset \overline{V} \subset U$.

<u>Proof</u>: Clearly $X - U$ is a closed set not containing x. Since (X, T) is a regular space, there exist disjoint sets $W, V \in T$ such that $X - U \subset W$ and $x \in V$. Hence $X - W \subset U$ and $x \in X - W$. Then $V \subset X - W \subset U$. Since $X - W$ is closed $\overline{X - W} = X - W$. So $\overline{V} \subset \overline{X - W} = X - W \subset U$. Therefore $\overline{V} \subset U$.

<u>Theorem 8</u>: Let (X, T) be a T_1 topological space, and let $a \in X$. Then $\{a\}$ is closed in X.

<u>Proof</u>: Let $b \in X$ with $a \neq b$. Then there exists a set $O_b \in T$ containing b and not a.
Thus $X - \{a\} = \cup\{O_b \mid b \neq a, b \in X\} \in T$. Thus $\{a\}$ is closed.

<u>Theorem 9</u>: Let (X, T) be a topological space such that if $a \in X$, then $\{a\}$ is closed in X. Then (X, T) is a T_1-space.

<u>Proof</u>: Let $b \in X$ with $b \neq a$. Then $X - \{a\}$ is an open set in X containing b and not containing a. Also, $X - \{b\}$ is an open set in X containing a but not containing b. Thus (X, T) is a T_1-space.

<u>Definition 16</u>: Let (X, T) be a topological space. Then (X, T) is said to be a <u>normal</u> space provided if A and B are disjoint closed sets in X, then there exists sets $U, V \in T$ such that $A \subset U$, $B \subset V$, and $U \cap V = \phi$.

<u>Theorem 10</u>: Let (X, T) be a normal T_1 topological space. Then (X, T) is a regular space.

<u>Proof</u>: Let A be a closed set in X. Since (X, T) is a T_1-space, if $x \in X - A$, then $\{x\}$ is closed. Since (X, T) is a normal space, there exist disjoint sets in T about A and about $\{x\}$. Thus (X, T) is a regular space.

<u>Example 2</u>: Let X be the reals, and let T be the collection of subsets of X to which U belongs provided either $U = \phi$, $U = X$, or there is an $x \in X$ such that $U = (x, \infty)$. Then (X, T) is a normal topological space which is not regular.

<u>Proof</u>: Let $U, V \in T$. Then clearly if $U = \phi$ or $V = \phi$, then $U \cap V \in T$. Also if $U = X$ or $V = X$, then $U \cap V \in T$. So suppose there exist real numbers a and b such that $U = (a, \infty)$ and $V = (b, \infty)$. Now if $a \leq b$, then $U \cap V = V$, and if $b \leq a$, then $U \cap V = U$. So $U \cap V \in T$. Let $\S = \{U_\alpha \mid \alpha \in \Gamma\}$ such that if $\alpha \in \Gamma$, then $U_\alpha \in T$. If for some $\alpha \in \Gamma$, $U_\alpha = X$, then clearly $\cup\{U_\alpha \mid \alpha \in \Gamma\} \in T$. So let $\alpha \in \Gamma$ so that $U_\alpha \neq \phi$. Then there exists a real number a such that $U_\alpha = (a, \infty)$. Let x be the greatest lower bound of $\{a \mid U_\alpha = (a, \infty), U_\alpha \neq \phi, \text{ and } U_\alpha \in \Gamma\}$. Then $\cup\{U_\alpha \mid \alpha \in \Gamma\} = (x, \infty)$ and $\cup\{U_\alpha \mid \alpha \in \Gamma\} \in T$. Thus T is a topology for X. Now, (X, T) must be a normal space since no two nonempty closed sets are disjoint. Also since $(0, \infty)$ is in T, $(-\infty, 0]$ is a closed subset of (X, T). Clearly $1 \notin (-\infty, 0]$. But every element of T which contains $(-\infty, 0]$ also contains 1. Hence (X, T) is not regular.

<u>Theorem 11</u>: Let (X, T) be a compact regular topological space. Then (X, T) is a normal space.

<u>Proof</u>: Let A and B be disjoint closed subsets of X. Let $a \in A$. Then since (X, T) is a regular space, there exist disjoint open sets U_a and V_a such that $a \in U_a$ and $B \subset V_a$. Let $\S = \{U_a \cap B \mid a \in A\}$. Since A is a closed subset of a compact space, A is compact. Since $\S$ is an open cover of A, there exists a finite subcover, $V_{a_1} \cap A, V_{a_2} \cap A, \ldots, V_{a_n} \cap A$, of $\S$. Then $A \subset \bigcup_{i=1}^{n} U_{a_i}$ and $B \subset \bigcap_{i=1}^{n} V_{a_i}$ and $\bigcup_{i=1}^{n} U_{a_i}$ and $\bigcap_{i=1}^{n} V_{a_i}$ are disjoint open subsets of X. Hence (X, T) is a normal space.

CHAPTER II

KC, US, AND K SPACES

Definition 17: A topological space (X, T) is said to be a KC space provided every compact set is closed.

Theorem 12: Let (X, T) be a T_2 topological space. Then (X, T) is a KC space.

Proof: Let A be a compact subset of X. By Theorem 5, A must be closed.

Theorem 13: Let (X, T) be a KC topological space. Then (X, T) is a T_1-space.

Proof: Let $x \in X$. Let Σ be an open cover of $\{x\}$ by elements of T. Since $\{x\}$ is an element of some $U \in \Sigma$, then U is a finite subcover of Σ covering $\{x\}$. So $\{x\}$ is compact. Then from Theorem 12, $\{x\}$ is closed. Then by Theorem 9, (X, T) is a T_1-space.

Definition 18: Let (X, T) be a topological space. A sequence $\langle x_n \rangle$ in X is said to converge to $x \in X$ provided, if U is a neighborhood of x, then there exists some positive integer N such that if n is a positive integer greater than or equal to N, then $x_n \in U$.

Definition 19: Let (X, T) be a topological space. Then (X, T) is said to be a US space provided if $\langle x_n \rangle$ is a sequence in X and $\langle x_n \rangle$ converges to x and y, then $y = x$.

Theorem 14: Let (X, T) be a US topological space. Then (X, T) is a T_1-space.

Proof: Let $a, b \in X$, with $a \neq b$. Define the sequence $\langle x_n \rangle$ by $x_n = a$ for each positive integer n. Then clearly $\langle x_n \rangle$ converges to a, and since X is a US space $\langle x_n \rangle$ does not converge to b. Thus there exists an open set U such that $b \in U$, and if N is a positive integer, then there exists a positive integer $n \geq N$ so that $x_n \notin U$. Thus $a \notin U$. Similarly, there is an open set V containing a but not b. Hence (X, T) is a T_1-space.

Example 3: Let X be the reals. Let T be the collection of subsets of X whose complements are countable, together with ϕ and X. Then (X, T) is a KC space which is not a T_2-space.

Proof: Let A be an uncountable subset of X. Let $\langle x_n \rangle$ be a sequence of distinct elements in A. For each positive integer N, define $U_N = X - \{x_j \mid j \geq n\}$. Then $\S = \{U_N \mid N \text{ is a positive integer}\}$ is an open cover of A by T_A-open sets which has no finite subcover. Thus A is not compact. So the only compact subsets of X must be countable. Finally, since every countable subset of X is closed, (X, T) is a KC space. And since no two nonempty open sets are disjoint, (X, T) is not a T_2-space.

Example 4: Let X be the reals. Let T be the collection of subsets of X whose complements are finite, together with ϕ and X. Then (X, T) is a T_1-space which is not a US space.

Proof: Let $a \in X$. Then $\{a\}$ is closed in X. By Theorem 9, (X, T) is a T_1-space. Let $\langle x_n \rangle$ be a sequence of distinct points in

X. Let $x \in X$. Let $U \in T$ with $x \in U$. Then U contains all but finitely many terms of $\langle x_n \rangle$. Thus $\langle x_n \rangle$ converges to x. Hence $\langle x_n \rangle$ converges to every element of X, and thus (X, T) is not a US space.

<u>Theorem 15</u>: Let (X, T) be a KC topological space. Then (X, T) is a US space.

<u>Proof</u>: Let $\langle x_n \rangle$ be a sequence in X which converges to a and to b, with $a \neq b$. Let $C = \{a\} \cup \{x_n \mid n \text{ is a positive integer}\}$. Let $\S$ be an open cover of C. Then there is an element $U \in \S$ such that $a \in U$. Since $\langle x_n \rangle$ converges to a, there is a positive integer N such that if $n \geq N$, then $x_N \in U$. For each positive integer K, $1 \leq K \leq N-1$, let $U_K \in \S$ such that $x_K \in U_K$. Then $\{U_K \quad 1 \leq K \leq N\} \cup \{U\}$ is a finite subcover of $\S$. Hence C is compact. However C is not closed since $b \in \overline{C}$ and $b \notin C$. But this is impossible since (X, T) is a KC space. Hence (X, T) is a US space.

<u>Definition 20</u>: Let (X, T) be a topological space. Then (X, T) is said to be <u>first countable</u> provided if $x \in X$, then there exists a countable neighborhood base at x.

<u>Theorem 16</u>: Let (X, T) be a first countable topological space. Let A be a subset of X, and let $x \in X$. If $x \in \overline{A}$, then there is a sequence $\langle x_n \rangle$ of points in A such that $\langle x_n \rangle$ converges to x.

<u>Proof</u>: Let $B_x = \{U_n \mid n \text{ is a positive integer}\}$ be a countable neighborhood base at x. For each positive integer N, define $V_N = \bigcap_{K=1}^{N} U_k$. For each positive integer N since $V_N \cap A \neq \phi$,

there exists an $x_N \in V_N \cap A$. Then $\langle x_N \rangle$ is a sequence of elements of A which converges to x.

<u>Theorem 17</u>: Let (X, T) be a first countable US topological space. Then (X, T) is a T_2-space.

<u>Proof</u>: Let $a, b \in X$, with $a \neq b$. Let $A = \{U_n \mid n \text{ is a positive integer}\}$ and $B = \{V_n \mid n \text{ is a positive integer}\}$ be countable neighborhood bases about a and b respectively. For each positive integer n, let $A_n = \bigcap_{K=1}^{n} U_K$ and let $B_n = \bigcap_{K=1}^{n} V_K$. Suppose $A_n \cap B_n \neq \phi$ for each positive integer n. Then let $x_n \in A_n \cap B_n$. Then $\langle x_n \rangle$ is a sequence of elements in X. Let U be an open set containing a. Then there is a positive integer m such that $a \in A_m \subset U$. Let $n \geq m$. Then $x_n \in A_n \cap B_n \subset A_n \subset A_m \subset U$. Thus $\langle x_n \rangle$ converges to a. Similarly $\langle x_n \rangle$ converges to b. But this is impossible because (X, T) is a US space. Hence there is a positive integer N such that $A_N \cap B_N = \phi$. Thus (X, T) is a T_2-space.

<u>Definition 21</u>: Let (X, T) be a topological space. Then (X, T) is said to be <u>locally compact</u>, provided, if U is a neighborhood about $x \in X$, then there exists a neighborhood V in X such that $x \in V \subset U$, and (V, T_V) is a compact subspace of (X, T).

<u>Lemma 1</u>: Let (X, T) be a topological space, and let $x \in X$. If for each $x \in X$ there is a neighborhood base at x consisting of sets closed in X, then (X, T) is a regular space.

<u>Proof</u>: Let A be a closed set in X, and let $x \in X - A$. Then $X - A$ is a neighborhood of x. So there exists a closed neighborhood

B in X such that $x \in B \subset X - A$. Let $O_B = \cup \{U_B \mid U_B \in T \text{ and } U_B \subset B\}$. Now O_B and $X - B$ are disjoint sets in T with $x \in O_B$ and $A \subset X - B$. Thus (X, T) is a regular topological space.

Theorem 18: Let (X, T) be a locally compact KC topological space. Then (X, T) is a T_2-space.

Proof: Let $x, y \in X$ with $x \neq y$. Let U_x be a neighborhood about x. Since (X, T) is locally compact, there exists a neighborhood V_x in X such that $x \in V_x \subset U_x$ and (U_x, T_{V_x}) is a compact subspace of (X, T). Let $\Psi_x = \{V_x \mid x \in U_x \text{ and } (U_x, T_{V_x}) \text{ is compact}\}$. Then by Definition 3, Ψ_x is a neighborhood base at x. Since (X, T) is a KC space, if $V_x \in \Psi_x$, then V_x is closed in X. So Ψ_x is a neighborhood base at x of closed sets in X. So by the lemma, (X, T) is a regular space. Also by Theorem 13, since (X, T) is a KC space, then (X, T) is a T_1-space. Thus $\{y\}$ is closed in X with $x \notin \{y\}$. Now since (X, T) is a regular space $\{x\}$ and $\{y\}$ can be separated by disjoint sets in T. Therefore (X, T) is a T_2-space.

Lemma 2: Let (X, T) be a topological space. Let n be a positive integer. Let $\{A_i \mid 1 \leq i \leq n\}$ be a finite collection of compact subsets of X. Then $\cup\{A_i \mid 1 \leq i \leq n\}$ is a compact subset of X.

Proof: Let Ψ be an open cover of $\cup\{A_i \mid 1 \leq i \leq n\}$ by $T_{\cup\{A_i \mid 1 \leq i \leq n\}}$ open sets. Let $A_m \in \{A_i \mid 1 \leq i \leq n\}$. Define Ψ_m to be $\{U \cap A_m \mid U \in \Psi\}$. Then Ψ_m is an open cover of A_m

and since A_m is compact, there exists a finite subcollection $U_{m_1} \cap A$, $U_{m_2} \cap A$, . . . , $U_{m_N} \cap A$ which covers A_m. Let $\S_m = \{U_{m_i} \mid 1 \leq i \leq N\}$. Then $\cup \{\S_m \mid 1 \leq m \leq N\}$ is a finite subcover of Ψ. Thus $\cup \{A_i \mid 1 \leq i \leq N\}$ is a compact subset of X.

Definition 22: Let (X, T) be a locally compact, non-compact, T_2-space, and let p be an element not in X. Let $X^* = X \cup \{p\}$, and let T_{X^*} be the set to which U belongs provided either $U \in T$ or $p \in U$ and $X^* - U$ is a compact subset of (X, T).

Theorem 19: Let (X, T) be a locally compact, non-compact, T_2-space, and let p be an element not in X. Then (X^*, T_{X^*}) is a topological space.

Proof: Clearly $\phi, X \in T_{X^*}$. Let $\{U_i \mid 1 \leq i \leq N\} \subset T_{X^*}$ where $U_i \in T$ for $1 \leq i \leq N$. Then $\cap \{U_i \mid 1 \leq i \leq N \in T\}$ and thus $\cap \{U_i \mid 1 \leq i \leq N\} \in T_{X^*}$. Let $\{U_\alpha \mid \alpha \in \Gamma\} \subset T_{X^*}$, where $U_\alpha \in T$ for each $\alpha \in \Gamma$. Then $\cup\{U_\alpha \mid \alpha \in \Gamma\} \in T$, and thus $\cup\{U_\alpha \mid \alpha \in \Gamma\} \in T_{X^*}$. Now let $\{U_i \mid 1 \leq i \leq N\} \subset T_{X^*}$ where $p \in \cap \{U_i \mid 1 \leq i \leq n\}$. Then
$X - \cap\{U_i \mid 1 \leq i \leq N\} = \cup \{X - U_i \mid 1 \leq i \leq N\}$ by DeMorgan's Laws. So $\cup\{X - U_i \mid 1 \leq i \leq N\}$ is the finite union of compact sets, which by Lemma 2 is compact. Since $X - \cap \{U_i \mid 1 \leq i \leq N\}$ is compact, $\cap \{U_i \mid 1 \leq i \leq N\}$ is in T_{X^*}. Now let $p \in \cup\{U_\alpha \mid \alpha \in \Gamma\}$ where $U_\alpha \in T_{X^*}$ for $\alpha \in \Gamma$. Then $p \in U_\beta$ for some $\beta \in \Gamma$. Now
$X - \cup\{U_\alpha \mid \alpha \in \Gamma\} = \cap\{X - U_\alpha \mid \alpha \in \Gamma\} \subset X - U_\beta$. Then $\cap\{X - U_\alpha \mid \alpha \in \Gamma\}$ is closed and compact in X. Finally (X^*, T_{X^*}) is a topological space.

<u>Definition 23</u>: Let (X, T) be a locally compact, noncompact, T_2 topological space and let p be an element not in X. Let $X^* = X \cup p$ and let T_{X^*} be the set to which U belongs provided either $U \in T$ or $p \in U$ and $X^* - U$ is a compact subset of X. Then (X^*, T_{X^*}) is called <u>the one-point compactification of</u> (X,T).

<u>Theorem 20</u>: Let (X,T) be a locally compact, noncompact, T_2 topological space. Then (X^*, T_{X^*}) is a compact space.

<u>Proof</u>: Let $\P$ be an open cover of X^*. Then there is an element $U_0 \in \P$ such that $p \in U_0$. Now, $X - U_0$ is a compact subset of X. Let $\Psi = \{U - \{p\} \mid U \in \P\}$. Let U be an element of T_{X^*} which contains p. Then $X^* - U$ is a compact subset of X and hence closed. Thus, $U - \{p\} = X - (X^* - U)$ is an open subset of X. Thus Ψ is an open cover of $X - U_0$. Since $X - U_0$ is compact, there exists a finite subcollection $V_1, \ldots, V_N$ of Ψ which covers $X - U_0$. If $V_i \in \P$, let $W_i = V_i$ and if $V_i \notin \P$, let $W_i = V_i \cup \{p\}$. Then $\{W_i \mid 1 \leq i \leq N\} \cup \{X - U_0\}$ is a finite subcover of $\P$ which covers X^*.

<u>Theorem 21</u>: Let (X, T) be a locally compact, noncompact, T_2, KC topological space. Then (X^*, T_{X^*}) is a US space.

<u>Proof</u>: Let $\langle x_n \rangle$ be a sequence which converges to $a, b \in X^*$, where $a, b \neq p$. Let $C = \{a\} \cup \{x_n \mid n$ is a positive integer$\}$. Then let $\P$ be an open cover of C by T_{X^*} open sets. Then there is an element $U \in \P$ such that $a \in U$. Since $\langle x_n \rangle$ converges to a, there is a positive integer N such that if $n \geq N$, then $x_n \in U$. For each positive integer K, $1 \leq K \leq N-1$, let $U_K \in \P$ such that $x_K \in U_K$. Then

$\{U_K \mid 1 \leq K \leq N\} \cup \{U\}$ is a finite subcover of ¶. Hence C is compact in X . Also, since $p \notin C$, C is a compact subset of X. However, C is not closed in X since $b \in \overline{C}$ and $b \notin C$. But this is impossible since (X, T) is a KC space. Hence (X^*, T_{X^*}) is a US space.

Definition 24: Let (X, T) be a topological space. Then (X, T) is said to be a K space provided, if any subset S, such that $S \cap K$ is closed for all closed and compact sets K, is itself closed in X.

Theorem 22: Let (X, T) be a locally compact, T_2, non-compact, and KC space. Then (X^*, T_{X^*}) is a KC space.

Proof: Let S be a compact subset of X^*. If $S \subset X$, then S is compact in X and hence closed in X, because X is a KC space, and therefore closed in X^*. Let p be the element of X^* not in X. If $p \in S$, let $F = S - \{p\}$. Now let K be a closed compact subset of X. Then K is closed in X^* and so $S \cap K$ is compact in X^*. Also $S \cap K$ is compact in X since $S \cap K$ is a subset of S. Since X is a KC space, $S \cap K$ is closed in X, and so $F \cap K = S \cap K$ is closed in X. Since X is a K space, it follows that F is closed in X. Then S is closed in X because $X^* - S = X - F$ is open and hence open in X^*.

Theorem 23: Let (X, T) be a locally compact, noncompact, T_2, KC topological space, and let (X^*, T_{X^*}) be a KC space. Then (X, T) is a K space.

Proof: Let $S \subset X$ such that $S \cap K$ is closed for every closed and compact $K \subset X$. Let p be the element of X^* not in X. Let

$\P$ be an open cover of F. There is a $U \in \P$ such that $p \in U$. Now $X - U$ is closed and compact in X, so $X - U$ is closed and compact in X. Thus $(X - U) \cap S$ is closed in X, and $(X - U) \cap S \subset X - U$, which is compact and hence $(x - U) \cap S$ is compact in X. Since $(X - U) \cap S = S - U$, then $S - U$ is compact. Let $\Psi = \{V - \{p\} \mid V \in \P\}$. Then Ψ is a covering of $S - U$ by T-open sets which must have a finite subcover, $O_1, O_2, \ldots, O_N$ where $O_i \in \Psi$. If $1 \leq i \leq N$, then if $O_i \in \P$, let $W_i = O_i$. Otherwise, $W_i = O_i \cup \{p\}$. Let $\Sigma = \{W_i \mid 1 \leq i \leq N\} \cup \{U\}$. Now Σ is a covering of F which is a finite subcover of $\P$. Thus F is compact in X^*. So F is closed in X^* since X^* is KC. But $S = F \cap X$. So S is closed in X.

Definition 25: Let A be a set and for each $\alpha \in A$, let (X_α, T_α) be a topological space. Then the Cartesian product of $\{X_\alpha \mid \alpha \in A\}$, denoted by $\times\{X_\alpha \mid \alpha \in A\}$, is the set of all functions $x\colon A \to \{\cup X_\alpha \mid \alpha \in A\}$ such that if $\alpha \in A$, then $x(\alpha) \in X_\alpha$. If $\beta \in A$, then $\pi_\beta\colon \times\{X_\alpha \mid \alpha \in A\} \to X_\beta$ is the function defined by if $x \in \times\{X_\alpha \mid \alpha \in A\}$, then $\pi_\beta(x) = x(\beta)$ and is called the β^{th} projection map. The product topology for $\times\{X_\alpha \mid \alpha \in A\}$ is the topology generated by $\{\pi_\alpha^{-1}(U_\alpha) \mid \alpha \in A, U_\alpha \in T_\alpha\}$.

Definition 26: Let (X, T) be a locally compact, T_2-space. Let I be the closed unit interval under the subspace topology induced by the usual topology on the reals. Let C be the set of all continuous functions from X into I. For each $c \in C$, let $I_c = I$. Let $Y = \times\{I_c \mid c \in C\}$. Then Y is compact and T_2 under the product topology [3, Theorem 17.8, page 120]. Define $e\colon X \to Y$ by if

$x \in X$, then $e(x)(f) = f(x)$. The Stone-Cech compactification of X is the closure of X in Y under the usual subspace topology [3, Definition 19.4, page 137].

Definition 27: A space (X, T) is pseudo-finite provided its only compact subsets are finite.

Theorem 24: Every T_1, pseudo-finite, K-space is discrete.

Proof: Let (X, T) be a T_1, pseudo-finite, K-space. Let $p \in X$. Let K be a closed, compact subset of X. Then K is finite. If $p \notin K$, then $(X - \{p\}) \cap K = K$ and hence $(X - \{p\}) \cap K$ is closed. If $p \in K$, then since (X, T) is T_1 and K is finite, p is not a limit point of K. Thus $K - \{p\}$ is closed. But $(X - \{p\}) \cap K = K - \{p\}$ and hence once again $(X - \{p\}) \cap K$ is closed. Since (X, T) is a K-space, $X - \{p\}$ is closed and hence $\{p\}$ is open. Hence T is the discrete topology for X.

Example 5: Let N be the positive integers with the discrete topology. Let Y be the Stone-Cech compactification of N and let $p \in Y - N$. Let $X = N \cup \{p\}$ with the subspace topology T received from the topology on Y. Then X is a compact US space but not a KC space.

Proof: If S is any infinite subset of N, neither S nor $S \cup \{p\}$ is closed in Y and thus neither is compact. Thus (X, T) is pseudo-finite. By Theorem 24, since (X, T) is not discrete, (X, T) is not a K-space. Since (X, T) is a T_2-space, (X, T) is a KC space. Thus by Theorem 21, X^* is a US space. But since

(X, T) is not a K-space, it then follows from Theorem 23 that X^* is not a KC space.

SUMMARY

In this paper, it has been shown that every T_2-space is a KC space is a US space is a T_1-space and that no implication is reversible.

It has been shown that a locally compact KC space is a T_2-space.

One interesting unsolved problem is whether a locally compact US space must be a T_2-space.

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